On Perturbation Spectra of N-flation

Yun-Song Piao
College of Physical Sciences, Graduate School of Chinese Academy of Sciences, YuQuan Road 19A, Beijing 100049, China

In this note we study the adiabatic perturbation spectrum of N-flation with power law potential. We show that the scalar spectrum of N-flation is generally redder than that of its corresponding single field. The result obtained for that with unequal massive fields is consistent with the recent numerical investigation of Kim and Liddle.

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The multi-field inflation is promising in the building of inflation models, because it relaxes many bounds on single field inflation models. Generally many fields can work cooperatively to drive a period of inflation by assisted inflation mechanism proposed by Liddle et.al [1], see also subsequent works [2] and [3] for that with a spectrum of masses states, even if any one of those field is not able separately to sustain inflation[1]. There have been some interesting examples of multi-field inflation, see Ref. [5], in which exponentially large number of fields was required for a feasible theoretic realization of inflation. Recently, Dimopoulos et.al [6] have shown that the many axion fields predicted by string vacuum can be combined and lead to a radiatively stable inflation, called N-flation, which may be an attractive embedding of inflation in string theory.

In the simplest case where all fields have the same mass, there is an attractor corresponding to the radial motion in field space. Dimopoulos et.al assumed that all fields started with the same initial condition, which equals to place them on this radial trajectory. Thus they showed that the adiabatic perturbation has the same spectral index as in the single field case. This has been confirmed further in Ref. [7]. However, in the N-flation setup there are large numbers of axion fields, and all fields have different masses, which can be very densely spaced. Thus a detailed exploration of the dynamics and the adiabatic perturbation spectra in the unequal mass case seems indispensable. The results were firstly mentioned in Ref. [8]. The detailed study was made by Easther and McAllister [9], however, restricted to quite specific choices of initial conditions for the fields. Recently, Kim and Liddle [10] have carried out numerical investigations in which the random initial conditions were used. They found that the scalar spectral index has significant dependence on the parameters of model, but when the number of fields becomes enough large, the spectral index predicted will be independent of initial conditions and enter into a long plateau. In this note firstly we will educe a formula on the spectral index of adiabatic perturbation of N-flation with power law potential, and then with this formula we will try to give some semi-analytical studies on the adiabatic perturbation spectrum of N-flation with the unequal massive fields..

Though for a set of uncoupled fields with general power law potential $\Lambda_i(\phi_i/\mu_i)^n$, where the subscript 'i' denotes the relevant quantities with the ith field and n is the same for all fields, there is no attractor solution corresponding to the radial motion in field space, assisted inflation remains. The reason is that the collection effect from multi-field enhances the friction for individual field. Thus we will use the slow-roll approximation for all formula in the following. The efoldings number is

$$N \simeq -\frac{8\pi}{m_p^2}\sum_i \int_{\phi_i}^{\phi_i^e} \frac{V_i}{V_i'}d\phi_i \simeq \frac{4\pi}{nm_p^2}\sum_i \phi_i^2\,, \qquad (1)$$

where the lower limits ϕ_i^e of the integrals correspond to the end of inflation and have been neglected, which can be reasonable for interesting cases.

The curvature perturbation of N-flation can be evaluated by using Sasaki-Stewart formalism [11], see also earlier Ref. [12], in which the curvature perturbation on comoving slices is expressed as the perturbation of efoldings number, which in turn is given in terms of the inflaton perturbation on flat slices after horizon crossing. For a set of above uncoupled fields, the spectral index is given by

$$\begin{aligned} n_s - 1 \;\simeq\; & -\frac{m_p^2}{8\pi}\frac{\sum_i (V_i')^2}{V^2} - \frac{m_p^2}{4\pi\sum_i(\frac{V_i}{V_i'})^2} \\ & + \frac{m_p^2}{4\pi V}\frac{\sum_i(\frac{V_i}{V_i'})^2 V_i''}{\sum_j(\frac{V_j}{V_j'})^2} \\ =\; & -\frac{1}{N}\left[1+\frac{n}{2}\cdot\frac{\sum_i(\frac{\Lambda_i}{\mu_i^n})^2\phi_i^{2(n-1)}\sum_i\phi_i^2}{(\sum_i\frac{\Lambda_i}{\mu_i^n}\phi_i^n)^2}\right], \qquad (2) \end{aligned}$$

where (1) has been used in the last line and $V\equiv\sum_i V_i$. This expression depends only on the quantities at horizon crossing, however, it in fact is a good approximation, as was discussed in Ref. [10]. Generally there also exist orthogonal isocurvature perturbations [13, 14], see Ref. [15] for a review, which might or might not become important depending on the evolution after inflation. However, here we will only focus on the adiabatic perturbation for our purpose. Now we define $g_i \equiv (\frac{\Lambda_i}{\mu_i^n})/(\frac{\Lambda_1}{\mu_1^n})$ and $f_i\equiv\phi_i/\phi_1$,

[1] However, note that in Ref. [4], a realistic inflation model based on MSSM has been proposed.

and then institute them into (2). Thus n_s can be written as

$$n_s - 1 \simeq -\frac{n+2}{2N}\left[1 + \frac{nR(g_i, f_i)}{n+2}\right], \quad (3)$$

where

$$R(g_i, f_i) = \frac{\sum_{i<j} f_j^2 f_i^2 \left(g_j f_j^{(n-2)} - g_i f_i^{(n-2)}\right)^2}{(\sum_i g_i f_i^n)^2}. \quad (4)$$

This is our main result in this note. The first term in the right side of Eq. (3) is just the result of single field inflation with power law potential, while the second term can be regarded as a correction induced by the variance between different fields, i.e. the differences between their parameters and their initial conditions. The interesting point of this formula is that it can be seen very easily that the correction term is always positive, which indicates that the scalar spectrum of N-flation is generally redder than that of its corresponding single field. Note also that this formula is invariance under the exchange between i and j, which means that the spectral index is independent of the detailed array of serial number of fields.

For $n = 2$, which corresponds to N-flation with massive fields $m_i^2\phi_i^2/2$, we can obtain from Eqs. (3) and (4)

$$n_s - 1 \simeq -\frac{2}{N}\left[1 + \frac{R(g_i, f_i)}{2}\right], \quad (5)$$

where

$$R(g_i, f_i) = \frac{\sum_{i<j} f_j^2 f_i^2 (g_j - g_i)^2}{\sum_i (g_i f_i^2)^2}, \quad (6)$$

where $g_i = m_i^2/m_1^2$. Immediately we can see that when $g_j = g_i$, we have $R(g_i, f_i) = 0$, independent of the value of each field at the time when the perturbation spectrum is calculated. Thus when the masses of all fields are equal, the scalar spectrum of N-flation will be the same as that of its corresponding single field, independent of the initial conditions of fields.

The adiabatic perturbation spectrum of N-flation with unequal massive fields is generally redder than that of its corresponding single field has been mentioned in Ref. [8], however, it seems that it is not obvious to obtain such a conclusion and also how the spectrum is dependent of unequal masses was not illustrated. The relevant result was also obtained in Ref. [9], however, restricted to quite specific choices of initial conditions and mass distribution for the fields. But in our Eqs. (3) and (4), one can straightly see that the spectrum is redder than that of its corresponding single field, not only for massive field but for general power law potential, which is not dependent of the initial conditions and the distribution of relative parameters, such as the masses and couples of fields.

We discuss the perturbation spectrum of N-flation with unequal massive fields with Eqs. (5) and (6) in the following. Firstly note that when there are only two massive fields, the result of Lyth and Riotto [16]

$$n_s - 1 \simeq -\frac{1}{N}\left[2 + \frac{f^2(g-1)^2}{(1+gf^2)^2}\right] \quad (7)$$

can be obtained, where $g = m_2^2/m_1^2$ and $f = \phi_2/\phi_1$. n_s depends explicitly on g and f, the ratio of the values of two fields at the time with N efoldings number, This makes us hardly obtain the definite value of n_s.

For the number $\mathcal{N} \gg 1$ of fields, we take the mass spectrum as $m_i^2 = m_1^2 \exp((i-1)/\sigma)$, where $i = 1, 2, ...\mathcal{N}$, as in Ref. [10], and σ gives the density of fields per logarithmic mass interval. In the slow-roll regime the field equation is $3h\dot{\phi}_i + m_i^2\phi_i \simeq 0$, thus the fields obey the conditions

$$f_i^2 = \frac{\phi_i^2}{\phi_1^2} \simeq \left(\frac{\phi_1}{\phi_{1,0}}\right)^{2(\frac{m_i^2}{m_1^2}-1)} \frac{\phi_{i,0}^2}{\phi_{1,0}^2}, \quad (8)$$

where the subscript '0' denotes the initial value of field.

When $\mathcal{N} \ll \sigma$, the masses of all fields can be approximately written as $m_i^2 = m_1^2[1 + (i-1)/\sigma]$. Thus $g_i - 1 = (i-1)/\sigma \ll 1$, which leads to

$$g_j - g_i = \frac{j-i}{\sigma} < \frac{\mathcal{N}}{\sigma} \ll 1 \quad (9)$$

and in the meantime since $g_i \simeq 1$, Eq.(8) can be reduced to $f_i^2 \simeq \phi_{i,0}^2/\phi_{1,0}^2$. These indicate that each term in numerator of Eq.(6) is $\ll 1$ while each term in denominator is ~ 1, except there is a large hierarchy between $\phi_{i,0}$ and $\phi_{1,0}$. Thus we can deduce $R(g_i, f_i) \ll 1$. Note that adding the number of fields does not effect this conclusion, which can be seen as follows. For a fixed large $\mathcal{N}$, the term number of numerator is $\mathcal{N}(\mathcal{N}-1)/2 \sim \mathcal{N}^2$, thus the value of numerator is approximately $\mathcal{N}^2\delta$, where $\delta \ll 1$, while the value of denominator is about $\mathcal{N}^2\mathcal{O}(1)$. Therefore for the case that the number of fields is far smaller that σ, the spectral index is basically the same as that of single field. The main reason is that in this case the increasing by degrees of mass of field is negligible, which makes the result similar to that of N-flation with same massive fields.

When $\mathcal{N} \simeq \sigma$, the difference of masses between different fields begins to become important. Thus from Eq. (6), the spectral index will shift towards the red direction. The limit case is $\mathcal{N} \gg \sigma$. In this case after some value i_c, we can have $g_i = \exp[(i_c - 1)/\sigma] \gg 1$. Note further that generally $\phi_1/\phi_0 < 1$. Thus from Eq.(8), for an enough large mass, we have $g_i f_i^2 \ll 1$, since the increasing of g_i leads to the exponential suppression of the value of f_i^2. This suggests that after i approaches some value i_c, the contribution of $g_i f_i^2$ to the denominator of (6) may be neglected and thus $\mathcal{N} - i_c$ terms after it, and similar result may be applied for those of numerator. Due to cut-off of the contributions from the fields with enough large mass, the spread of spectrum towards red direction is certainly not arbitrary large. Further it can be expected that the shift generally constrained in a quite small region, as has been numerically shown in Ref. [10].

For $n = 4$, which corresponds to N-flation with $\lambda_i\phi_i^4$ fields, we can obtain from Eqs. (3) and (4)

$$n_s - 1 \simeq -\frac{3}{N}\left[1 + \frac{2R(g_i, f_i)}{3}\right], \tag{10}$$

where

$$R(g_i, f_i) = \frac{\sum_{i<j} f_j^2 f_i^2 \left(g_j f_j^2 - g_i f_i^2\right)^2}{\sum_i \left(g_i f_i^4\right)^2}, \tag{11}$$

where $g_i = \lambda_i/\lambda_1$. The single field $\lambda\phi^4$ inflation has been ruled out by WMAP combined with SDSS [17], see also [18] and [19] for discussions on the bounds of WMAP on the inflationary model space. Thus with more fields, the spectrum will be generally more red, which is certainly less interesting. However, it can be noted from (11) that in the N-flation with $\lambda_i\phi_i^4$, even if all couples λ_i are equal, $R(g_i, f_i)$ also dose not vanish, since the terms in the bracket of numerator are not only dependent of λ_i but the field value ϕ_i, which is distinctly different from that of $m_i^2\phi_i^2/2$. The cases for $n > 4$ are similar to that of $n = 4$.

The tensor/scalar ratio r is also an important inflation quantity for observation, which as well as n_s makes up of the $r - n_s$ plane [20], in which different classes of inflation modes are placed in different regions. In Ref. [10], it has been shown that the tensor/scalar ratio in N-flation model with massive fields depends only on the efoldings number and is independent of the number $\mathcal{N}$ of fields, their masses m_i^2 and initial conditions, and always has the same value as that of its corresponding single field. In fact this result has been noticed in Ref. [8], and is also valid for the N-flation with general power law potential $\Lambda_i(\phi_i/\mu_i)^n$ discussed here.

In summary, we educe the formula (3) of the spectral index of adiabatic perturbation of N-flation with power law potential. This formula indicates that the scalar spectrum of N-flation is generally redder than that of its corresponding single field. Then with this formula we discuss the adiabatic perturbation spectrum of N-flation with the unequal massive fields. We found that the result is consistent with the numerical investigation of Kim and Liddle [10].

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